

# On the mixing of a low Reynolds number biological jet with a quiescent outer bathing solution

By **J. RICHARD GOLDGRABEN**

Polytechnic Institute of Brooklyn, New York

AND **SHELDON WEINBAUM**

City College of the City University, New York

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Existing analyses of the very low Reynolds number Stokes jet, e.g. Birkhoff & Zarantonello (1957) and Förste (1963), have been confined to a single-component fluid satisfying the usual zero-slip boundary conditions at a solid surface. In contrast, the low Reynolds number biological jets which emerge from the pores and channels at the secreting surfaces of numerous human, animal and insect organs entail the movement of water and solute subject to novel boundary conditions that arise from the local osmotic driving forces at the secreting surfaces. These boundary conditions introduce a nonlinear coupling between the fluid momentum and solute conservation equations. This paper first discusses the fundamental dimensionless groups and length scales that characterize these biological jet flows and then examines in detail, using the technique of matched inner and outer expansions, one flow situation important in epithelial membrane transport, namely, the two-dimensional mixing of an inhomogeneous jet with a quiescent outer bathing solution which is bounded by a semi-permeable membrane in the plane of the exit. The paper concludes with a discussion of other physiologically relevant problems which arise from different orderings of the length scales, and different overall geometrical configurations and flow conditions.

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## 1. Introduction

Laminar and turbulent jet mixing problems at high Reynolds number have long been of interest to fluid mechanicians because of their numerous applications in both compressible and incompressible flows. Low Reynolds number jets, on the other hand, have had relatively few applications and those studies that have been conducted have been for highly idealized flow situations. One area of important application for low Reynolds number jets that has been largely overlooked by fluid mechanicians is biological flow. Unfortunately, the idealized models of Stokes jets that have been analysed have little relevance for biological flows since these models treat a pure fluid rather than a two-species continuum of water and solute and do not consider biologically meaningful boundary conditions. A rather extensive variety of jets mixing with both stagnant and well-stirred outer bathing solutions will need to be examined in the future. The present paper examines in

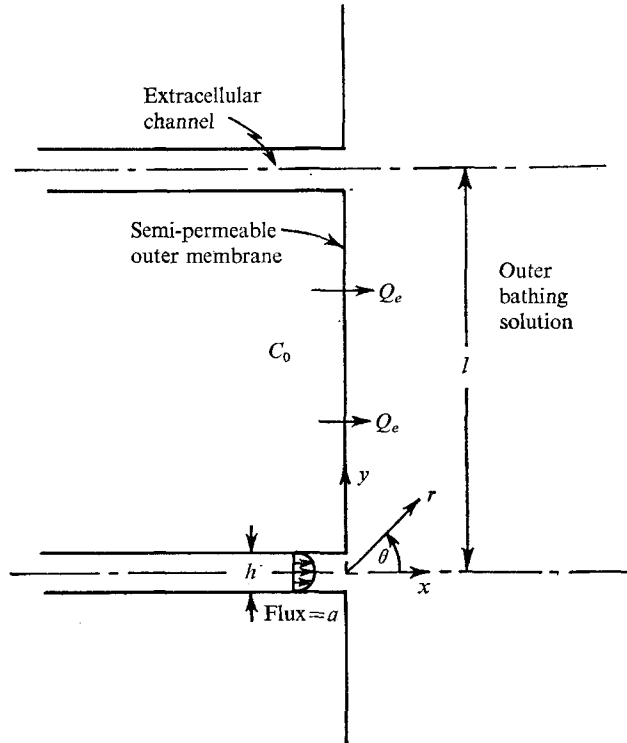


FIGURE 1. Schematic diagram of exit mixing region adjacent to epithelial cell layer.

detail only one particularly simple biological jet flow problem, namely the mixing of an inhomogeneous effluent from a single two-dimensional channel with a quiescent outer bathing solution occupying a half-plane which is bounded by a semi-permeable osmotic membrane in the plane of the exit that allows the passage of water but not solute. The particular application of interest to the authors is the mixing that ensues at the exit of the extracellular channels that characterize epithelial cell layers, see figure 1.

The qualitative behaviour of biological jets is much more subtle than that for single-component Stokes jets since it involves the interaction between several different transport processes that can occur on the same or different length scales depending on the relative values of the five different dimensionless groups  $a/\nu$ ,  $a/D$ ,  $M_s C_0/\rho$ ,  $\rho a/P_w C_0 h$  and  $l/h$  that characterize the particular flow problem. Here  $a$  is the total volumetric efflux per unit channel depth and is defined as

$$\int_{-\frac{1}{2}h}^{\frac{1}{2}h} u_x dy, \quad \text{where } u_x \text{ is the streamwise velocity component at } x = 0;$$

$\nu$  is the kinematic viscosity;  $D$  is the mass diffusivity coefficient for solute in water;  $\rho$  is the total mass density;  $M_s$  is the solute molecular weight;  $C_0$  is the solute molar concentration in the interior of the cells which comprise the cell layer. The cell interior is usually assumed to be well mixed and hence the concentration  $C_0$  is assumed to be uniform throughout the cell and is taken to be

the characteristic reference concentration for the problem.  $P_w$  is the water permeability coefficient of the semi-permeable outer membrane of figure 1 and has units of g/s cm<sup>2</sup> per mole/ml;  $h$  is the channel height and  $l$  the channel spacing if an array of channels is considered.  $a/\nu$  is therefore an exit Reynolds number,  $a/D$  a ratio of convective to diffusional solute flux,  $M_s C_0/\rho$  a solute-solvent density ratio,  $\rho a/P_w C_0 h$  a ratio of the channel efflux to the local passive water flux at the boundary and  $l/h$  a dimensionless channel spacing. Typical values of these dimensionless numbers for epithelial membrane problems are

$$\frac{a}{\nu} \approx O(10^{-6}), \quad \frac{a}{D} \approx O(10^{-3}), \quad \frac{M_s C_0}{\rho} \approx O(10^{-2}), \quad \frac{\rho a}{P_w C_0 h} \approx O(10^2), \quad \frac{l}{h} \approx O(10^3).$$

In general two characteristic lengths other than  $l$  and  $h$  are possible for low  $Re$  jets with  $a/\nu \ll 1$ :  $D/U_0$ , where  $U_0$  is the channel centre-line velocity at the exit, and  $\rho a/P_w C_0$ . The length scale  $D/U_0$  is a measure of the distance from the source at which the convective and diffusional terms in the solute conservation equation become of comparable magnitude and thus does not appear if  $a/D \leq O(1)$  since there is then no inner length scale in which the solute transport is convection dominated. The length scale  $\rho a/P_w C_0$  is a measure of the distance from the channel exit at which the osmotic water flux at the boundary becomes comparable with the local convective flux due to the exit source. From the typical values given in the last paragraph one concludes that there are three important characteristic lengths for the two-dimensional jets emanating from the extracellular channels of epithelial membranes, and that these lengths are ordered as  $h \ll \rho a/P_w C_0 \ll l$ .

The present analysis shows that on the inner scale  $h$  the osmotic movement of water across the outer membrane ( $Q_e$  in figure 1) is a higher order effect and the primary transport process is the diffusional relaxation of the inhomogeneity in the exit concentration profile. On the intermediate scale  $\rho a/P_w C_0$  the osmotic water flux  $Q_e$  at the outer boundary is important and causes first-order changes in the streamline pattern and higher order convective corrections in the concentration distribution. On the largest length scale  $l$  the flow problem is essentially that of a slow viscous flow with a periodic water flux boundary condition in the plane of the channel exits, assuming that the latter are equally spaced. The present paper will be concerned with obtaining matched asymptotic solutions which are uniformly valid on the first two length scales. Thus, the solutions on the length scales  $h$  and  $\rho a/P_w C_0$  will be referred to as the near and far fields respectively throughout the remainder of the paper. This two-region problem can also obviously be viewed as the exit mixing problem for a single isolated biological channel flow.

In concluding this section we shall briefly outline the physiological context of the exit jet problem for epithelial cell layers. The interested reader is referred to Stein (1967) and Bittar (1970) for a general survey of the physiological function of the various epithelial cell layers. These cell layers line the surfaces of many human, animal and insect organs such as the ciliary body of the eye, gall bladder, frog skin, avian salt gland and renal collecting tubule, to mention a few, and have the primary function of transporting water and solutes between the serosal fluids in the capillary beds and the mucosal fluids in the lumen or cavities of



pressure distributions which develop within both open and closed extracellular channels under the influence of hydrostatic, local and transmembrane osmotic driving forces. Local osmotic driving forces are produced when metabolically related solute ion pumps create localized regions of high solute concentration within the extracellular channel. Water then flows passively from the cell interior into the channel across the essentially semi-permeable lateral membrane. The solution for the channel flow just described provides the initial condition at the channel exit on the concentration and water flux for the jet mixing problem treated herein.

Sections 2 and 3 contain, respectively, the far- and near-field solutions. Numerical results are presented in § 4 and a brief discussion of additional areas in which biological jet theory might be applied is presented in § 5.

## 2. Far-field solution

The governing equations for the far field are the Navier-Stokes momentum equation written in terms of the vorticity  $\boldsymbol{\omega} = \nabla \times \mathbf{u}$  to eliminate the pressure term, a continuity equation for the incompressible fluid and a solute conservation equation. In vector form these are, respectively,

$$\mathbf{u} \cdot \nabla \boldsymbol{\omega} = \nu \nabla^2 \boldsymbol{\omega}, \tag{2.1}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{2.2}$$

$$-D\nabla^2 C + \mathbf{u} \cdot \nabla C = 0, \tag{2.3}$$

where  $\mathbf{u}$  is the bulk fluid velocity and  $C$  is the solute concentration.

On the length scale of the far field the Stokes jet emerging from the channel exit appears as a point source of strength  $a$ , located at the origin of the polar co-ordinate system  $(r, \theta)$  shown in figure 1. The radial and circumferential velocity components are then both of the order of the convective velocity from the source. From the work of Birkhoff & Zarantonello (1957), the radial velocity in a two-dimensional Stokes jet which satisfies a no-slip boundary condition is of order  $a/r$ . Equations (2.1)–(2.3) are therefore non-dimensionalized by defining the dimensionless variables

$$u^* = \frac{u}{a/r}, \quad v^* = \frac{v}{a/r}, \quad R = \frac{r}{k}, \quad C^* = \frac{C}{C_0}, \quad f = \frac{\psi}{a}, \tag{2.4}$$

where  $u$  and  $v$  are the velocity components in the  $r$  and  $\theta$  directions respectively and are related to the two-dimensional stream function  $\psi$  by

$$u = \frac{1}{r} \frac{\partial \psi}{\partial \theta}, \quad v = -\frac{\partial \psi}{\partial r}, \tag{2.5}$$

$k$  is used to represent the outer length scale  $\rho a / P_w C_0$  and  $f$  is a dimensionless stream function. All quantities defined in (2.4) are of order unity.

Substitution of (2.4) into (2.5) yields

$$u = \partial f / \partial \theta, \quad v = -R \partial f / \partial R, \tag{2.6}$$

where asterisks in (2.6) and in all subsequent discussion have been dropped. Equation (2.1) can be written in terms of the dimensionless velocity components as

$$\begin{aligned} (a/\nu) \{u[R^2v_{RR} - rv_R + v_{\theta\theta} + 2u_\theta] + v[Rv_{R\theta} - u_{\theta\theta}]\} \\ = R^3v_{RRR} - R^2v_{RR} + Rv_R + 2Rv_{\theta\theta R} - 4v_{\theta\theta} - 4v_\theta - u_{\theta\theta\theta}. \end{aligned} \quad (2.7)$$

The subscripts in (2.7) indicate partial differentiation in the usual manner.

Since  $a/\nu \ll 1$  and  $R = O(1)$  one obtains after substituting (2.6) into (2.7) the biharmonic equation in polar co-ordinates:

$$R^4f_{RRRR} + 2R^3f_{RRR} - R^2f_{RR} + Rf_R + 2R^2f_{RR\theta\theta} - 2Rf_{R\theta\theta} + 4f_{\theta\theta} + f_{\theta\theta\theta} = 0. \quad (2.8)$$

Substitution of (2.5) and (2.6) into (2.3) yields

$$-R^2C_{RR} - RC_R - C_{\theta\theta} + \xi Rf_\theta C_R - \xi Rf_R C_\theta, \quad (2.9)$$

where  $\xi = a/D$ .

We seek solutions of (2.8) and (2.9) subject to the boundary conditions that the solutions are symmetric about  $\theta = 0$ , the tangential velocity component vanishes at the outer membrane  $\theta = \pm \frac{1}{2}\pi$  and that the mass flux of water crossing this boundary is proportional to the osmotic driving force. This latter condition is written in dimensional terms as

$$v|_{\frac{1}{2}\pi} = \frac{P_w}{\rho} [C - C_0]_{\frac{1}{2}\pi}.$$

On using the definitions in (2.4) and dropping asterisks this becomes

$$v|_{\frac{1}{2}\pi} = -R[C - 1]_{\frac{1}{2}\pi}. \quad (2.10)$$

A compatibility condition, defining membrane semi-permeability, must also be imposed on the concentration gradient  $\partial C/\partial\theta$  and the water flux  $v$  at the outer membrane so that the net solute flux due to convection and diffusion is zero. The diffusional flux of solute  $J_{s\theta}$  in the  $\theta$  direction is assumed to be linearly related to the concentration gradient:

$$J_{s\theta} = -M_s D \frac{C_0}{k} \frac{1}{R} \frac{\partial C}{\partial\theta}. \quad (2.11)$$

When the sum of (2.11) and the convective solute flux produced by the velocity  $v$  in (2.10) is equated to zero, one obtains

$$[\partial C/\partial\theta]_{\theta=\frac{1}{2}\pi} = -[\xi RC(C - 1)]_{\theta=\frac{1}{2}\pi}. \quad (2.12)$$

Boundary conditions (2.10) and (2.12) do not by themselves uniquely determine the far-field velocity and concentration solutions for an isolated jet. As one might anticipate, some of the arbitrary constants of integration are left unspecified. These constants must be determined by matching with a near-field solution which satisfies appropriate boundary conditions at the channel exit as the near-field radial co-ordinate approaches infinity or  $R$  approaches zero. Similarly, if one were considering the interaction between adjacent jets on the length scale  $l$ , one would need to match again, as  $R$  approaches infinity, to determine the unknown constants that would appear in the solution for this next larger length scale. The locally valid series solutions which we shall now

construct to describe the behaviour on the length scale  $k$ , however, are not suitable for matching with a larger length scale solution since they are only valid in the range  $0 < R < 1$  and do not apply as  $R$  approaches infinity.

The difficulty in solving (2.8) and (2.9) arises primarily through the coupling introduced in the boundary condition (2.10). We therefore formulate the solutions for the stream function and solute concentration as co-ordinate expansions in  $R$ , where the zeroth-order solutions are those for which this boundary condition is homogeneous.

We assume that

$$f = f^{(0)} + Rf^{(1)} + O(R^2), \tag{2.13a}$$

$$C = C^{(0)} + RC^{(1)} + O(R^2), \tag{2.13b}$$

where  $f^{(i)}$  and  $C^{(i)}$  are functions of  $\theta$  only.

Substitution of (2.13a, b) into (2.8) and (2.9) and equating terms of like order in  $R$  yields to zeroth order

$$4f_{\theta\theta}^{(0)} + f_{\theta\theta\theta\theta}^{(0)} = 0, \quad C_{\theta\theta}^{(0)} = 0. \tag{2.14a, b}$$

To first order in  $R$

$$f_{\theta\theta\theta\theta}^{(1)} + 2f_{\theta\theta}^{(1)} + f^{(1)} = 0, \tag{2.15a}$$

$$C^{(1)} + C_{\theta\theta}^{(1)} - \xi C^{(1)} f_{\theta}^{(0)} + \xi f^{(1)} C_{\theta}^{(0)} = 0. \tag{2.15b}$$

With a similar substitution into the boundary conditions (2.10),  $v = 0$  to zeroth order in  $R$  and (2.12) produces a homogeneous boundary condition on  $C_{\theta}^{(0)}$ .

To first order in  $R$  equations (2.10) and (2.12) become

$$f^{(1)}|_{\theta=\frac{1}{2}\pi} = [C^{(0)} - 1]_{\theta=\frac{1}{2}\pi}, \tag{2.16a}$$

$$C_{\theta}^{(1)}|_{\theta=\frac{1}{2}\pi} = -[\xi C^{(0)}(C^{(0)} - 1)]_{\theta=\frac{1}{2}\pi}. \tag{2.16b}$$

The solution of (2.14a) satisfying the condition of symmetry at  $\theta = 0$  and no slip at  $\theta = \frac{1}{2}\pi$  is equivalent to the Stokes jet solution with radial streamlines as obtained by Birkhoff & Zarantonello (1957). In the variables defined by (2.4), this solution becomes

$$f^{(0)} = \frac{1}{2}\beta_1(\sin 2\theta + 2\theta), \tag{2.17a}$$

where  $\beta_1$  is a constant representing the channel source strength. The solution of (2.14b) satisfying the zeroth-order approximation to (2.12) and the symmetry condition is

$$C^{(0)} = \beta_2, \tag{2.17b}$$

where  $\beta_2$  is a constant to be determined by matching with the near-field solution. A first-order correction to the streamline pattern is produced by the passive water movement across the outer membrane and is obtained from (2.15a) after applying (2.16a) and the symmetry and no-slip boundary conditions:

$$f^{(1)} = (\beta_2 - 1) \sin \theta. \tag{2.18}$$

When results (2.17a, b) and (2.18) are substituted into (2.15b) the governing equation for the first-order correction to the uniform concentration field becomes

$$C_{\theta\theta}^{(1)} + [(1 - \xi\beta_1) - \xi\beta_1 \cos 2\theta] C^{(1)} = 0. \tag{2.19}$$

This is the Mathieu equation and since  $\xi \ll 1$  it can be solved by a small parameter expansion of the form

$$C^{(1)} = C_1 + \xi\beta_1 C_2 + O(\xi\beta_1)^2. \quad (2.20)$$

On substituting (2.20) into (2.19) and equating like terms in  $\xi\beta_1$ ,

$$C_1'' + C_1 = 0, \quad (2.21)$$

$$C_2'' + C_2 = (1 + \cos 2\theta) C_1. \quad (2.22)$$

When the even solutions to (2.21) and (2.22) are substituted into (2.20) one obtains

$$C^{(1)} = A_1 \cos \theta + \xi\beta_1 [A_2 \cos \theta + \frac{3}{4}A_1 \theta \sin \theta - \frac{1}{16}A_1 \cos 3\theta] + O(\xi\beta_1)^2, \quad (2.23)$$

where  $A_1$  and  $A_2$  are constants of integration. These are evaluated by taking the derivative of (2.23) with respect to  $\theta$ , setting the result equal to the right-hand side of (2.16*b*) and equating terms in like powers of  $\xi$ . The first-order solution for the concentration is then found to be

$$C^{(1)} = \xi\beta_2(\beta_2 - 1) \cos \theta. \quad (2.24)$$

Substitution of (2.17*a*) and (2.18) into (2.13*a*), and of (2.17*b*) and (2.24) into (2.13*b*) yields finally

$$f = \frac{1}{2}\beta_1(\sin 2\theta + 2\theta) + R(\beta_2 - 1) \sin \theta + O(R^2), \quad (2.25a)$$

$$C = \beta_2 + R[\xi\beta_2(\beta_2 - 1) \cos \theta + O(\xi^2)] + O(R^2). \quad (2.25b)$$

One observes in (2.25*a, b*) that the passive water movement across the outer membrane produces a first-order correction of order  $R$  to the streamline pattern but only a higher order correction, of  $O(\xi R)$ , to the concentration field.

As already mentioned the solution (2.25*a*) for the far-field velocity approaches that for a two-dimensional Stokes jet from a point source as  $R$  approaches zero. A near-field velocity solution is therefore required which (i) behaves like the leading term in (2.25*a*) as the near-field radial co-ordinate becomes infinite, (ii) satisfies a Poiseuille parabolic velocity profile at the channel exit for

$$-\frac{1}{2}h \leq y \leq \frac{1}{2}h$$

and (iii) to lowest order in the expansion parameter  $h/k$  (see next section) obeys the usual zero-velocity boundary condition at the membrane surface in the near field. A solution to the Stokes flow equation (2.8) meeting all these requirements is given in Förste (1963). This result expressed in terms of dimensionless rectangular Cartesian velocity components is

$$\tilde{u}_x = -\frac{1}{\pi} \left[ (\tilde{x}^2 + \tilde{y}^2 - 1) \tan^{-1} \left( \frac{2\tilde{x}}{\tilde{x}^2 + \tilde{y}^2 - 1} \right) - 2\tilde{x} \right], \quad (2.26a)$$

$$\tilde{u}_y = -\frac{1}{\pi} \left[ -\tilde{x}^2 \ln \frac{\tilde{x}^2 + (\tilde{y} - 1)^2}{\tilde{x}^2 + (\tilde{y} + 1)^2} - 2\tilde{x}\tilde{y} \tan^{-1} \frac{2\tilde{x}}{\tilde{x}^2 + \tilde{y}^2 - 1} \right], \quad (2.26b)$$

where we have defined the dimensionless variables

$$\tilde{x} = \frac{2x}{h}, \quad \tilde{y} = \frac{2y}{h}, \quad \tilde{u}_x = \frac{u_x}{\frac{3}{2}a/h}, \quad \tilde{u}_y = \frac{u_y}{\frac{3}{2}a/h}.$$



If one now matches the  $x$  and  $y$  velocity components obtained from (2.25*a*), see equation (4.3), with the solution (2.26) as  $\tilde{x}$  and  $\tilde{y}$  approach infinity one finds that

$$\beta_1 = 1/\pi. \tag{2.27}$$

It is evident that the far-field concentration solution (2.25*b*) cannot satisfy an arbitrary initial concentration distribution at the channel exit. Thus one seeks a near-field solution for the concentration which satisfies appropriate initial conditions at the exit and which can be matched with the far-field solution in the limit as  $R$  approaches zero and the near-field radial co-ordinate approaches infinity. This solution and matching for the near-field concentration is considerably more complicated than that for the near-field velocity and is described separately in the next section.

### 3. Near-field concentration solution

The velocity and concentration distributions in the near field obey the governing equations (2.8) and (2.3) subject to boundary conditions (2.10) and (2.12) and to initial conditions at the channel exit. In addition, the near-field solutions for the velocity and concentration must behave like (2.25*a, b*) in the limit as the stretched near-field radial co-ordinate

$$r^* = \frac{r}{\frac{1}{2}h} = \frac{R}{h/2k} \tag{3.1}$$

approaches infinity and  $R$  approaches zero, or more exactly, values of  $O(h/k)$ . If one were now to express the governing equations and boundary conditions in terms of the stretched near-field radial co-ordinate and systematically construct solutions as power series in the small parameter  $h/k$ , one would find that the boundary condition (2.10) is homogeneous in the near field to  $O(h/k)$  (see equation (3.5)) and therefore that the near-field solution for the velocity (2.26) is valid to this order. Furthermore, if this representation for the near-field velocity were used in the solute conservation equation (2.3) the convective terms would be accurately described to  $O(h\xi/k)$ . The difficulty in proceeding in this manner is that (2.3), although linearized, is very difficult to solve because of the complex spatial dependence of the coefficients in the convective term.

We proceed, therefore, by introducing an approximate Oseen-type simplification for the convective term in (2.3) in the near field. To this end we shall simplify (2.3) by using a near-field representation of the far-field velocity solution (2.25*a*) as the near-field convective velocity. This velocity field provides an accurate representation for (2.3) for large values of the inner length scale and a qualitatively reasonable approximation as one approaches the exit region, where the approximate linearizing velocity reduces to that of a Stokes jet from a point source. One anticipates, therefore, that the resulting solution for the solute concentration field will provide a reasonable but not quantitatively accurate representation of the mixing process in the entire near field. Equations (2.25*a, b*), when expressed in terms of the stretched near-field co-ordinate, become

$$f = \frac{\beta_1}{2}(\sin 2\theta + 2\theta) + \frac{h}{k} \frac{r}{2}(\beta_2 - 1) \sin \theta + O\left[\left(\frac{h}{k}\right)^2\right], \tag{3.2*a*}$$

$$C = \beta_2 + \frac{hr}{k^2} [\xi \beta_2 (\beta_2 - 1) \cos \theta + O(\xi^2)] + O\left[\left(\frac{h}{k}\right)^2\right], \quad (3.2b)$$

where the asterisks on  $r$  have been dropped. Equation (3.2a) is our Oseen approximation for the velocity in the convective term in (2.3), while (3.2b) gives the behaviour of the far-field concentration in the region of overlapping validity. In order to match (3.2b) with the near-field concentration solution, we formulate the latter as a double asymptotic expansion in integral powers of  $h/k$  and  $\xi$ . We shall consider this asymptotic expansion to be a proper representation of the near field if matching can be accomplished and if boundary conditions can be satisfied to at least the order of approximation associated with the assumed near-field streamline pattern in (3.2a).

We let

$$C = C_0 + \frac{h}{k} C_1 + \left(\frac{h}{k}\right)^2 C_2 + O\left[\left(\frac{h}{k}\right)^3\right],$$

where

$$\left. \begin{aligned} C_0 &= C^{00} + \xi C^{01} + O(\xi^2), \\ C_1 &= C^{10} + \xi C^{11} + O(\xi^2), \\ C_2 &= C^{20} + \xi C^{21} + O(\xi^2). \end{aligned} \right\} \quad (3.3)$$

The ordering of the terms in (3.3) is, according to the magnitudes given in § 1,

$$1 \gg \frac{h}{k} \gg \xi \gg \left(\frac{h}{k}\right)^2 \gg \frac{h}{k} \xi \dots$$

On using (3.1), (3.2a) and the remaining dimensionless variables defined in (2.4), the solute conservation equation (2.3), the boundary conditions (2.10) and (2.12) for the near field become respectively

$$-r^2 C_{rr} - r C_r - C_{\theta\theta} + \xi r f_\theta C_r - \xi f_r C_\theta = 0, \quad (3.4)$$

$$v|_{\theta=\frac{1}{2}\pi} = -\frac{hr}{k^2} [C-1]_{\theta=\frac{1}{2}\pi}, \quad (3.5)$$

$$C_\theta|_{\theta=\frac{1}{2}\pi} = -\xi \frac{hr}{k^2} [C(C-1)]_{\theta=\frac{1}{2}\pi}. \quad (3.6)$$

If we define the domain of the near field as that bounded by  $\frac{1}{2}\pi \geq \theta \geq -\frac{1}{2}\pi$  and  $r \geq 1$ , we can avoid the difficulty associated with solving this boundary-value problem with mixed boundary conditions for concentration along the plane  $x = 0$ . Condition (3.5) shows, as mentioned previously, that the no-slip boundary condition on which (2.26) is based is valid to  $O(h/k)$  in the near field. For the concentration we require, in addition to (3.6), that  $C$  behave as (3.2b) for large  $r$ , that  $C = g(\theta)$  be an arbitrary but even function on  $r = 1$  and that the first term in (3.2a) account for the total volumetric flow emerging from the channel. In dimensionless form, the latter condition is satisfied by

$$\frac{\beta_1}{2} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\partial}{\partial \theta} (\sin 2\theta + 2\theta) d\theta = 1. \quad (3.7)$$

On integrating (3.7) one finds that

$$\beta_1 = 1/\pi. \tag{3.8}$$

Substitution of (3.3) into (3.4) and (3.6) yields, after equating terms in like powers of  $h/k$  and  $\xi$ , the equations and boundary conditions to be satisfied by the unknown functions  $C^{ij}$  in (3.3).

To order unity  $\nabla^2 C^{00} = 0, \quad C^{00}|_{\theta=\frac{1}{2}\pi} = 0,$  (3.9a, b)

where  $\nabla^2$  is the Laplacian in polar co-ordinates.

We also require that the initial condition on  $r = 1$  be satisfied identically to first order. That is

$$C^{00}|_{r=1} = g(\theta). \tag{3.9c}$$

The even solution to (3.9a) satisfying (3.9b, c) and the condition of boundedness is

$$C^{00} = A^{00} + \sum_{n=1}^{\infty} F_n^{00} r^{-2n} \cos 2n\theta, \tag{3.9d}$$

where  $A^{00}$  and  $F_n^{00}$  are the Fourier coefficients

$$\left. \begin{aligned} A^{00} &= \frac{2}{\pi} \int_0^{\frac{1}{2}\pi} g(\theta) d\theta, \\ F_n^{00} &= \frac{4}{\pi} \int_0^{\frac{1}{2}\pi} g(\theta) \cos 2n\theta d\theta \quad \text{for } n = 1, 2, 3, \dots \end{aligned} \right\} \tag{3.9e}$$

The solutions of order  $(h/k)^m$  for  $m = 1, 2, \dots$ , satisfy Laplace's equation and homogeneous boundary conditions on  $\theta = \frac{1}{2}\pi$  and  $r = 1$ . Therefore

$$C^{10} = C^{20} = C^{m0} = 0. \tag{3.10}$$

According to (3.10), one does not expect to find terms of order  $R^m$  appearing in the far-field concentration expansion.  $R$  will appear only as a mixed product with the parameter  $\xi$ .

To order  $\xi$  one must solve the equation

$$\nabla^2 C^{01} = r\beta_1(\cos 2\theta + 1) C_r^{00} \tag{3.11a}$$

subject to homogeneous boundary and initial conditions. Homogeneous solutions of (3.11a) of the form  $r^j \cos j\theta$  are not sufficiently general to satisfy the boundary and initial conditions. We must add to the series in integral powers of  $r$  an associated separable solution of order one. This solution is defined by  $\partial(r^j \cos j\theta)/\partial j$ . Associated separable solutions of any order  $n$  are seen to satisfy Laplace's equation since

$$\nabla^2 \left[ \frac{\partial^n}{\partial j^n} (r^j \cos j\theta) \right] = \frac{\partial^n}{\partial j^n} [\nabla^2 (r^j \cos j\theta)] = 0.$$

The complete solution to (3.11a), using the result of (3.8), is given by

$$C^{01} = A_1^{01} + \sum_{n=1}^{\infty} A_n^{01} r^{-2n} \cos 2n\theta + \sum_{n=1}^{\infty} \frac{F_n^{00}}{2\pi} r^{-2n} \times \left[ \frac{n}{2(2n+1)} \cos (2n+2)\theta - \frac{n}{2(2n-1)} \cos (2n-2)\theta + \ln r \cos 2n\theta \right], \tag{3.11b}$$

where

$$A_1^{01} = F_1^{00}/4\pi,$$

$$A_n^{01} = \frac{1}{4\pi} \left[ \frac{(n+1)F_{n+1}^{00}}{(2n+1)} - \frac{(n-1)F_{n-1}^{00}}{(2n-1)} \right] \quad \text{for } n = 2, 3, 4, \dots$$

If (3.9*d*), (3.10) and (3.11*b*) are substituted into (3.3) and the resulting series is rewritten in terms of the outer co-ordinate  $R$  by use of (3.1), a comparison of the three leading terms (of order unity,  $h/k$  and  $\xi$ ) with the corresponding terms in (2.25*b*) yields the matching condition

$$\beta_2 = A^{00} + \xi F_1^{00}/4\pi. \quad (3.12)$$

Results (3.10) and (3.12) shows that the functions  $C^{(i)}$  in (2.13*b*) are in fact power series expansions in the parameter  $\xi$ .

There is no justification for obtaining solutions of order  $(h/k)\xi$  because the assumed stream function (3.2*a*) does not satisfy boundary condition (3.5) to this order. The final approximate solution has, therefore, not included the higher order corrections due to the passive water movement across the outer boundary.

#### 4. Numerical results

Far-field velocity profiles  $u_x$  and  $u_y$  in the directions normal and parallel, respectively, to the outer membrane in figure 1 can be derived from (2.25*a*) and (2.6). Since in dimensional terms

$$u_x = u \cos \theta - v \sin \theta, \quad u_y = u \sin \theta + v \cos \theta,$$

one obtains on transformation to rectangular co-ordinates

$$u_x = \frac{2ax^3}{\pi(x^2 + y^2)^2} + \frac{a}{k}(\beta_2 - 1), \quad u_y = \frac{2ax^2y}{\pi(x^2 + y^2)^2}. \quad (4.1)$$

These equations are non-dimensionalized by defining the dimensionless variables

$$\eta = \frac{x}{k}, \quad \zeta = \frac{y}{k}, \quad u_\eta = \frac{u_x}{a/k}, \quad u_\zeta = \frac{u_y}{a/k}, \quad (4.2)$$

where  $a$  and  $k$  are as defined in §§ 1 and 2.

Substitution of (4.2) into (4.1) yields

$$u_\eta = \frac{2\eta^3}{\pi(\eta^2 + \zeta^2)^2} + (\beta_2 - 1), \quad u_\zeta = \frac{2\eta^2\zeta}{\pi(\zeta^2 + \eta^2)^2}. \quad (4.3a, b)$$

The normal velocity profiles of (4.3*a*) are plotted in figure 3 as curves of  $u_\eta - (\beta_2 - 1)$  versus  $\zeta$  at constant distances  $\eta$  away from the outer boundary. The scales indicated on the axes of figure 3 apply directly to the dimensionless co-ordinate distances  $\eta$  and  $\zeta$ . The ordinate scale also represents  $\frac{1}{10}$  of the magnitude of  $u_\eta - (\beta_2 - 1)$  when the profile heights are measured with respect to the dotted reference lines for  $\eta = 0.05, 0.1, 0.2, 0.5$  and  $1.0$ . The profiles of  $u_\eta$  for non-isotonic ( $\beta_2 \neq 1$ ) channel effluxes are simply those shown in figure 3 with a shift up or down of magnitude  $\beta_2 - 1$ . For a hypotonic effluent ( $\beta_2 < 1$ ) one observes from

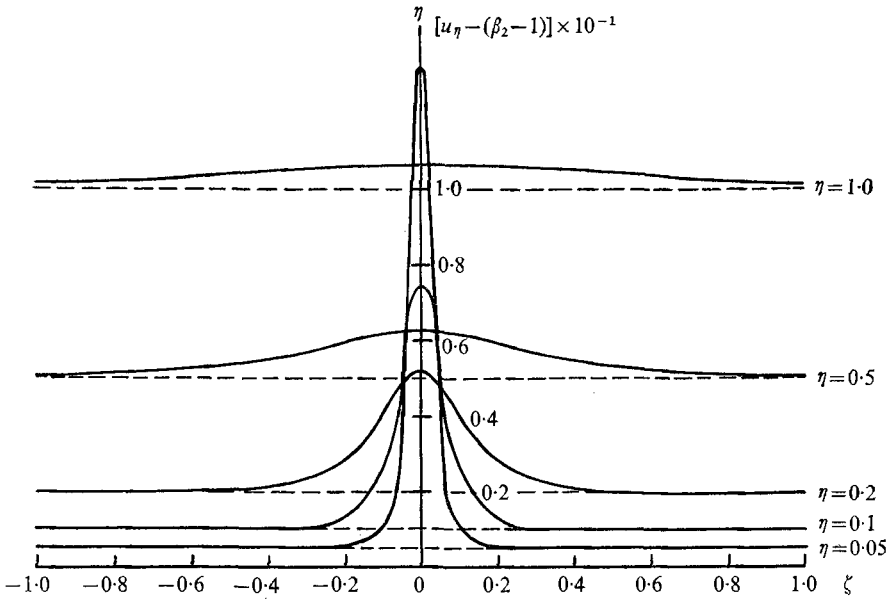


FIGURE 3. Variation in the far-field velocity profile  $u_\eta - (\beta_2 - 1)$  with increasing distance from the outer boundary. Ordinate scaling represents units of dimensionless distance and units of  $[u_\eta - (\beta_2 - 1)] \times 10^{-1}$ .

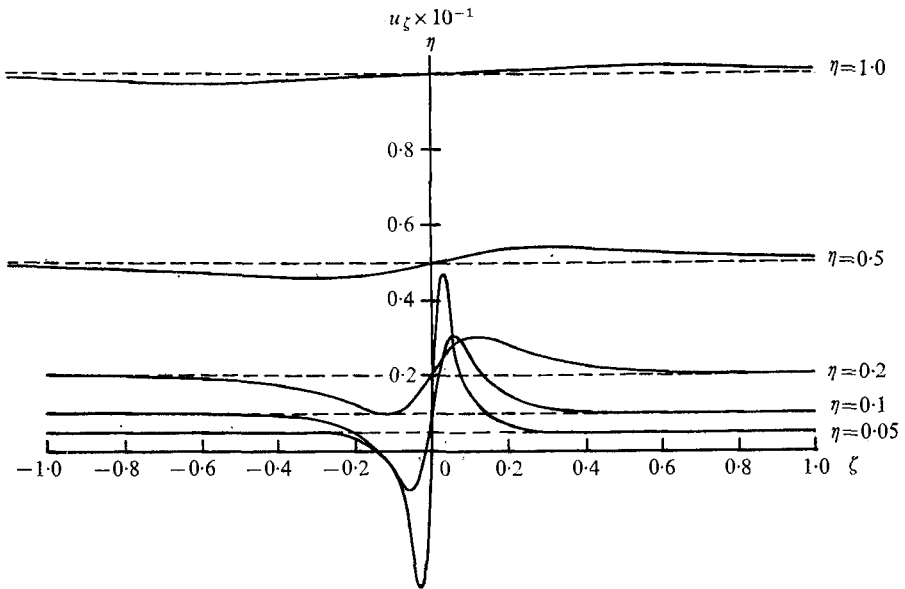


FIGURE 4. Variation in the far-field velocity profile  $u_\zeta$  with increasing distance from the outer boundary. Ordinate scaling represents units of dimensionless distance and units of  $u_\zeta \times 10^{-1}$ .

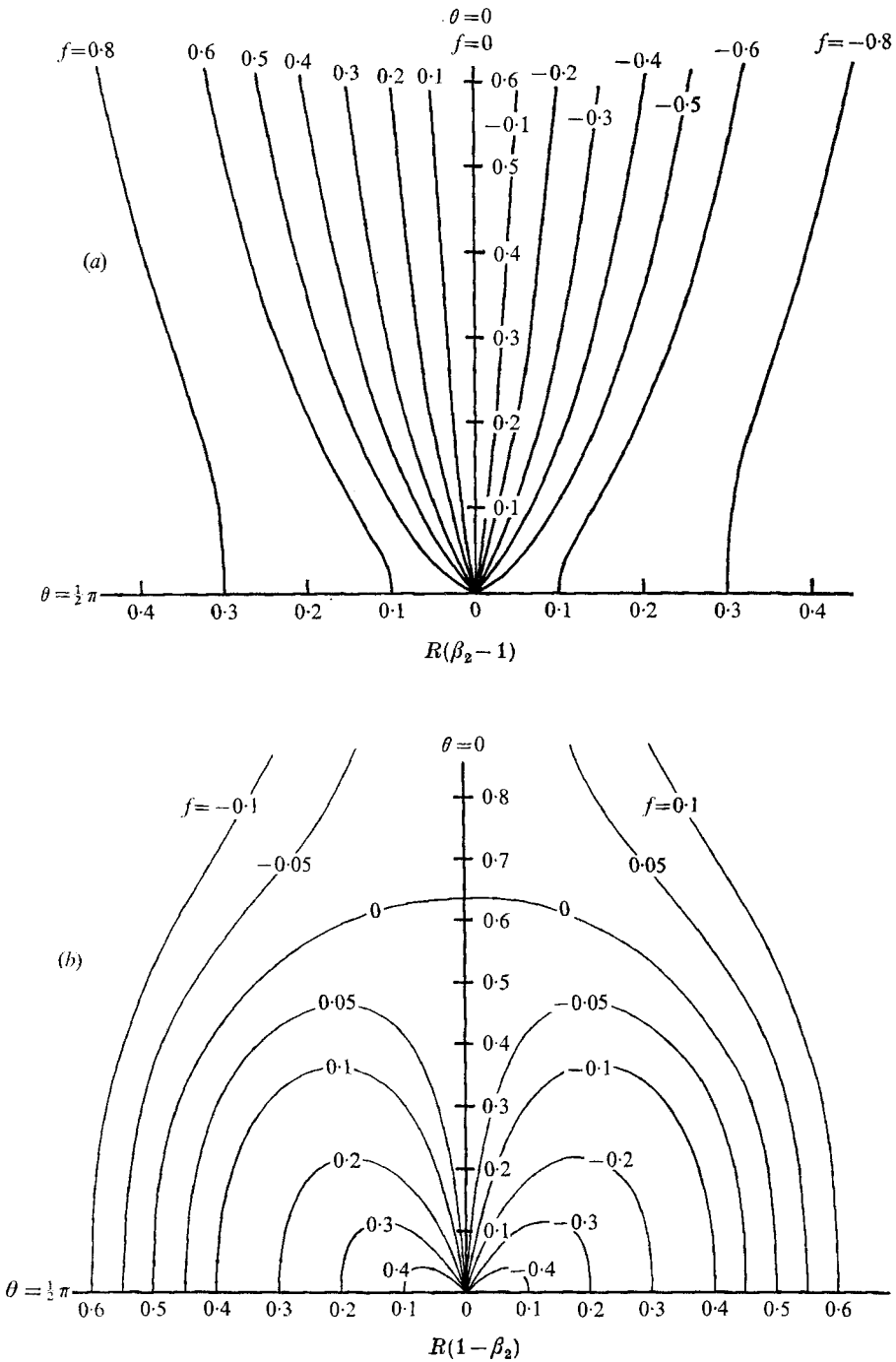


FIGURE 5. Dimensionless polar plot of the streamline pattern in the exit mixing region of epithelia with (a) hypertonic and (b) hypotonic effluent. The radial co-ordinates are  $R(\beta_2 - 1)$  and  $R(1 - \beta_2)$  respectively.

(4.3a) that a stagnation point exists along the channel centre-line ( $\zeta = 0$ ) at a distance  $\eta = 2/\pi(1 - \beta_2)$  from the channel exit.

The velocity profiles of (4.3b) are plotted in figure 4 with distance and velocity scales identical to those of figure 3.

The far-field streamline patterns resulting from different levels of exit concentration are most conveniently represented as streamlines in a polar coordinate system whose radial dimension is the quantity  $R(\beta_2 - 1)$  or  $R(1 - \beta_2)$  for hypertonic (i.e.  $\beta_2 > 1$ ) or hypotonic ( $\beta_2 < 1$ ) channel secretions respectively. Equation (2.25a), using (2.27), is plotted in this manner in figures 5(a) and (b). One observes that the radial streamline pattern ensues as  $R$  approaches zero or as the channel secretion approaches isotonicity ( $\beta_2 = 1$ ). In these limits the streamlines for  $f = +0.5$  become tangential to the boundary  $\theta = \pm \frac{1}{2}\pi$ .

For a hypotonic secretion, the streamline pattern in figure 5(b) clearly shows how reabsorption of fluid into the cell interior across the outer membrane requires that fluid be drawn from the channel exit as well as from the far field. The curved streamline  $f = 0$  delineates these two regions. The stagnation point occurs at  $\theta = 0$ ,  $R(1 - \beta_2) = 2/\pi$ .

The radial length scale in figure 5 also represents, to first order, the ratio of the volumetric flow crossing the outer boundary to the channel source strength. In dimensionless terms

$$\frac{Q_e}{a} = \int_0^R \frac{\partial f}{\partial R} dR \quad \text{on } \theta = \frac{1}{2}\pi. \quad (4.4)$$

Substitution of (2.25a) and (2.27) into (4.4) yields on integration

$$Q_e/a = R(\beta_2 - 1) + \dots, \quad (4.5)$$

where the positive direction of  $Q_e$  is taken as the positive  $x$  direction in figure 5(a).

Equation (2.25b) may be represented by lines of constant concentration ratio  $C/\beta_2$  in a polar co-ordinate system with radial length scale  $R\xi|(\beta_2 - 1)|$ . Then from (2.25b)

$$C/\beta_2 = 1 + R\xi(\beta_2 - 1) \cos \theta + \dots \quad (4.6)$$

Lines of constant concentration ratio in (4.6) lie parallel with the outer membrane, which has a concentration ratio of 1. With  $\beta_2$  greater or less than unity, the concentration ratio increases or decreases, respectively, with a uniform normal gradient of  $\pm 1$  using the above length scale.

The equilibration which is achieved in the near field by a non-homogeneous exit solute concentration distribution is readily demonstrated by assuming an exit distribution of the form

$$C(r = 1, \theta) = C_w + (C_m - C_w) \cos \theta, \quad (4.7)$$

where  $C_m$  and  $C_w$  are the dimensionless concentrations at the channel centre-line and wall respectively.

Substitution of (4.7) into (3.3), (3.9c, d, e), (3.11b) and (3.12) yields to order  $\xi$

$$\begin{aligned} \frac{C - C_w}{C_m - C_w} &= \frac{2}{\pi} + \frac{4}{3\pi} r^{-2} \cos 2\theta - \frac{4}{15\pi} r^{-4} \cos 4\theta \\ &+ \dots (-1)^{n+1} \frac{4}{(4n^2 - 1)\pi} r^{-2n} \cos (2n\theta) + O(\xi). \end{aligned} \quad (4.8)$$

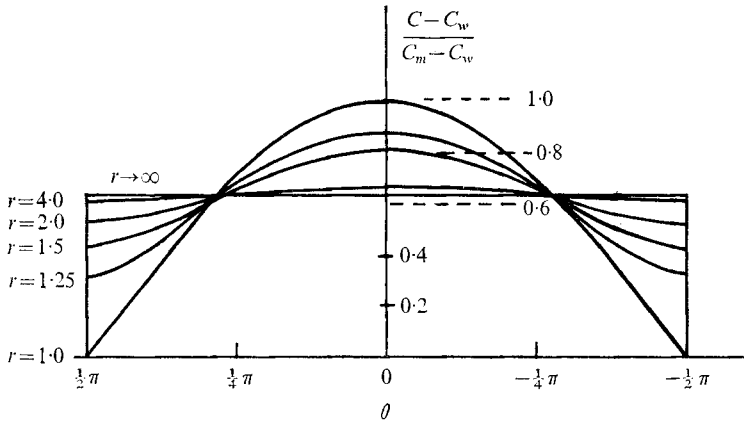


FIGURE 6. Variation in the near-field concentration distribution with increasing radial distance from the channel exit.

The variation in  $(C - C_w)/(C_m - C_w)$  along arcs of constant radius, as represented by (4.8), is shown in figure 6. At a radial distance of  $r = 4.0$ , the non-homogeneous concentration profile has come to within 5% of the uniform distribution which is achieved at large values of the near-field co-ordinate.

## 5. Conclusions

The development of theories for biological jets presents the fluid dynamicist with new and relevant areas for research. Several classes of problems can be identified according to the relative magnitudes of the dimensionless groups which characterize the flow. Most epithelia appear to transport fluids at rates which make the dimensionless groups  $a/\nu$  and  $a/D$  much less than unity. Biological jets are, then, low Reynolds number jets in which the length scales of interest are  $h$ ,  $\rho a/P_w C_0$ ,  $l$  and a radius of curvature  $\mathcal{R}$  of the epithelial surface.

For a relatively impermeable outer cell membrane,  $\rho a/P_w C_0$  becomes much larger than the channel spacing  $l$  and additional transport across this membrane can be neglected. The problem then involves the study of an array of jets emerging into the outer bathing solution. Work on a solution for this model has been started by the authors. New mathematical techniques are needed for the case in which  $\rho a/P_w C_0$  is of the order of the channel height. The work of Förste (1963) may, however, provide a basis for solution.

Since the epithelial tissue forms a closed surface which lines the various organs, the choice of a co-ordinate system most compatible with the physiological configuration is related to the magnitude of the dimensionless parameter  $\mathcal{R}/L$ , where  $L$  is one of the characteristic length scales described above. For  $\mathcal{R}/L \gg 1$  the model is best formulated as a half-plane in either rectangular or polar co-ordinates. For the gall bladder, as presented herein,  $\mathcal{R}/L \gg 1$  for each of the three characteristic length scales. In mammalian renal collecting tubules, see Grantham (1971),  $\mathcal{R}/L = O(1)$  when  $L$  represents the extracellular channel spacing. In this case a formulation in axisymmetric co-ordinates is called for.



The transport processes occurring in renal tubules and in the malpighian tubules and rectal lumen of insect excretory systems, see Wall (1971) and Oschman & Berridge (1971), present problems with non-stagnant bathing solutions. In organs such as these, the epithelia line the luminal surface of long cylindrical tubes which are closed at one end. The absorption or secretion of fluids then occurs across epithelia which are bounded on one side by the serosal fluid and on the luminal side by a bathing solution which has unidirectional flow velocities. The co-ordinate system chosen to represent this model again depends on  $R/L$ .

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